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A GAP RESULT OF SIMONS' TYPE FOR FREE BOUNDARY CMC-H SURFACES

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Abstract. We provide a gap theorem of Simons' type for free boundary minimal and constant mean curvature surfaces in the unit ball in 3-dimensional Euclidean space.

1. Introduction

Let B be the unit ball centered at the origin in $\mathbb{R}^3$. A compact immersed surface $\Sigma \subset B$ is said to be a free boundary cmc-H surface in the unit ball if the mean curvature H of Σ in $\mathbb{R}^3$ is constant and Σ meets ∂B orthogonally along its boundary $\partial\Sigma$. By using the first variation formula, a free boundary cmc-H surface Σ in the unit ball can be characterized as a critical point of area-functional for volume-preserving variations of Σ (resp. for variations of Σ if $H = 0$) satisfying $\partial\Sigma \subset \partial B$. In particular, it is called a free boundary minimal surface in the unit ball if $H = 0$.

The simplest examples of free boundary cmc-H surfaces are topological disks, equatorial disks and spherical caps. Nitsche [21] marked a turning point in research of free boundary cmc-H surfaces. He proved that the only free boundary cmc-H disks in the unit ball are equatorial disks and spherical caps. Ros-Souam [24] and Souam [28] extended this result to capillary cmc-H disks in the unit ball in $\mathbb{R}^3$ and free boundary

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cmc-H disks in geodesic balls in the 3-dimensional space forms, respectively. Recently, Fraser-Schoen [11] generalized Nitsche and Souam's results to free boundary minimal disks in geodesic balls of n-dimensional space forms.

After the pioneering works [10, 12] by Fraser-Schoen, there has been remarkable growth in the research of free boundary cmc-H surfaces. The critical catenoid is the next simplest example of a free boundary minimal surface in the unit ball. It is the specific scaling of a catenoid to satisfy the free boundary condition. Part of a Delaunay surface is a corresponding example for non-vanishing constant mean curvature. Until that time, these were all known embedded examples of cmc-H surfaces in the unit ball. Currently, a lot of examples of various topology are constructed in some ways (Steklov eigenvalues [12], the min-max construction [6, 15], and the desingularization method [9, 13, 14]). A characterization of the critical catenoid is one of the most interesting problems in this field. Some recent results in this regard are as follows: In [8, 27, 29], the authors computed the Morse index and the nullity of the critical catenoid. Symmetry for a free boundary minimal surface to become the critical catenoid is found in [16, 19]. In relation to the Steklov eigenvalue or Jacobi-Steklov eigenvalue, some characterization results are proved in [12, 30]. In [2, 5], a gap theorem is obtained, about which we will explain more below.

Bringing the focus back to works by Fraser-Schoen, they found that there is a strong relation between free boundary minimal surfaces in the unit ball in $\mathbb{R}^n$ and the first non-zero eigenvalue, which is called the first Steklov eigenvalue, of the Dirichlet-Neumann map. Such a relevance can also be found in $\mathbb{S}^n$: Closed minimal surfaces in $\mathbb{S}^n$ are related to the first eigenvalue of the Laplacian. Moreover, free boundary minimal surfaces in the unit ball in $\mathbb{R}^n$ is quite analogous to closed minimal surfaces in $\mathbb{S}^n$. For example, Nitsche's result corresponds to the following: The equator is the only immersed minimal surface in $\mathbb{S}^3$ of genus 0 (see [1]). Such similarity can also be confirmed in some of the recent results mentioned above.

In a celebrated paper [26], Simons proved the following theorem:

Theorem (Simons [26]). Let Σ be a closed minimal hypersurface in $\mathbb{S}^{n+1}$. Then

- either Σ is a totally geodesic;
- or $|A|^2 \equiv n$ on Σ;
- or $|A|^2(x) > n$ at some point $x \in \Sigma$,

where $|A|^2$ is the squared norm of the second fundamental form of Σ.

Remark that if $|A|^2 < n$, then $|A|^2 \equiv 0$ on Σ. Right after, Chern-do Carmo-Kobayashi [7] and Lawson [17] independently proved that if $|A|^2 \equiv n$ on Σ, then it is a Clifford minimal hypersurface, $\mathbb{S}^m\left(\sqrt{\frac{m}{n}}\right) \times \mathbb{S}^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$ for $1 \le m \le n-1$. Motivated by this result, Ambrozio-Nunes [2] proved the following gap theorem for free boundary minimal surfaces in the unit ball.

Theorem (Ambrozio-Nunes [2]). Let Σ be a compact free boundary minimal surface in the unit ball in $\mathbb{R}^3$. Assume that for every point $x \in \Sigma$,

$$|A|^2(x)\langle x, \nu(x)\rangle^2 \le 2,$$

where $\nu(x)$ denotes the unit normal vector at the point x and A denotes the second fundamental form of Σ. Then

1. either $|A|^2(x)\langle x, \nu(x)\rangle^2 \equiv 0$ and Σ is an equatorial flat disk;
2. or $|A|^2(x)\langle x, \nu(x)\rangle^2 = 2$ at some point $p \in \Sigma$ and Σ is a critical catenoid.

Barbosa-Cavalcante-Pereira [3] extended this result to free boundary cmc-H surfaces in the unit ball as follows.

Theorem (Barbosa-Cavalcante-Pereira [3]). Let Σ be a compact free boundary constant mean curvature surface in the unit ball in $\mathbb{R}^3$. Assume for all point $x \in \Sigma$,

$$|\mathring{A}|^2(x)\langle x, \nu(x)\rangle^2 \le \frac{1}{2}\left(2 + H\langle x, \nu(x)\rangle\right)^2,$$

where $\mathring{A}$ is the traceless second fundamental form of Σ. Then

1. either $|\mathring{A}|^2(x)\langle x, \nu(x)\rangle^2 \equiv 0$ and Σ is a spherical cap;
2. or equality occurs at some point $p \in \Sigma$ and Σ is part of a Delaunay surface.

Similar gap theorems hold in some 3-dimensional space different from $\mathbb{R}^3$: Li-Xiong [18] obtained a simiar gap theorem for free boundary minimal surfaces in geodesic balls of 3-dimensional space forms, and Min-Seo [20] for cmc-H surfaces in a strictly convex domain of a 3-dimensional Riemannian manifold.

The main purpose of this paper is to prove a gap theorem of Simons' type for a free boundary cmc-H surface in the unit ball in 3-dimensional Euclidean space as follows.

Theorem (see Theorem 3.3). Let Σ be a free boundary cmc-H surface in the unit ball in $\mathbb{R}^3$. Then

- either $|\mathring{A}|^2 \equiv 0$ and Σ is totally umbilical, i.e., Σ is an equatorial disk if $H = 0$ and a spherical cap if $H \neq 0$, respectively;
- or $|\mathring{A}|^2(x) \geq 2H^2 + 4 + 4H\langle x, \nu(x)\rangle$ at some point $x \in \Sigma$, furthermore, $|A|^2(x) > 4$ at some point $x \in \Sigma$ if $H = 0$.

Remark. The author found out about the work by Barbosa-Freitas-Melo-Vitório ([4], Theorem 9). They proved that Theorem 3.3 for a free boundary minimal submanifold Σ^n in $\mathbb{R}^{n+1}$, and the results are the same when $n = 2$. However the proofs have been obtained independently in different ways.

2. Preliminaries

In this section, we begin with some notions. Let Σ be an immersed surface in $\mathbb{R}^3$. The second fundamental form of Σ is a symmetric two tensor $A : T\Sigma \otimes T\Sigma \to C^\infty(\Sigma)$ defined to be, for all tangent vector fields $X, Y \in T\Sigma$,

$$A(X, Y) = \left\langle \overline{\nabla}_X Y, \nu \right\rangle,$$

where $\langle\ ,\ \rangle$ and $\overline{\nabla}$ denote the metric and the Riemannian connection on $\mathbb{R}^3$, respectively, and ν is the unit normal vector field of Σ. Denoted by H the mean curvature of Σ, that is given by

$$H = \frac{1}{2}\mathrm{trace}(A) = \frac{1}{2}\left(\lambda_1 + \lambda_2\right),$$

where λ_1, λ_2 are the principal curvatures of Σ. If H is constant on Σ, then Σ is said to be a constant mean curvature surface. To emphasize H we will call it *cmc-H surface*. In particular, if $H \equiv 0$ on Σ, then it is said to be a *minimal surface*.

Definition 2.1. Let $B = \{x \in \mathbb{R}^3 : \langle x, x\rangle = 1\}$ be the unit ball centered at the origin in $\mathbb{R}^3$. Let $\Sigma \subset B$ be an immersed surface in the unit ball B and $x : \Sigma \to B$ be the immersion of Σ such that $x(\mathrm{int}\Sigma) \subset \mathrm{int}B$ and $x(\partial\Sigma) \subset \partial B$. Then Σ is called a *free boundary cmc-H surface in the unit ball* if the following conditions hold:

- H is constant on Σ;
- Σ meets ∂B orthogonally along the boundary $\partial\Sigma$.

For later use, we introduce two results were shown by Ros-Vergasta in [25]. Note that they originally considered n-dimensional free boundary cmc-H hypersurfaces in the unit ball. For the sake of completeness, we describe the proofs for $n = 2$. Let Σ be a free boundary cmc-H surface in the unit ball. Let η be the inward pointing unit conormal vector field of $\partial\Sigma$ in Σ. Free boundary condition implies that $\eta = -x$ along $\partial\Sigma$. Then the geodesic curvature κ_g of $\partial\Sigma$ with respect to η is obtained as follows (see [25]):

$$\begin{aligned} \kappa_g &= \left\langle \overline{\nabla}_T T, \eta \right\rangle \\ &= \left\langle \overline{\nabla}_T(-\eta), T \right\rangle \\ &= \left\langle \overline{\nabla}_T x, T \right\rangle \\ &= \langle T, T \rangle \\ &= 1, \end{aligned}$$

where T denotes the unit tangent vector field of $\partial\Sigma$. Here, the third equality holds because x is the position vector.

Lemma 2.2. *Let Σ be a free boundary cmc-H surface in the unit ball. Then $\partial\Sigma$ is a line of curvature of Σ and the geodesic curvature of $\partial\Sigma$ in Σ with respect to the inward pointing unit conormal vector field is equal to* 1.

Remark that the first part of Lemma 2.2 is obtained directly from Joachimsthal's theorem.

Lemma 2.3 (First Minkowski formula). *Let Σ be a free boundary cmc-H surface in the unit ball. Then*

$$L(\partial\Sigma) = 2\left(A(\Sigma) + \int_\Sigma H\langle x, \nu\rangle \, dA\right),$$

where $L(\partial\Sigma)$ and $A(\Sigma)$ denote the length of $\partial\Sigma$ and the area of Σ, respectively.

Proof. Let r be the distance function measured from the origin in $\mathbb{R}^3$. Restricted to Σ, taking the Laplacian, we have

$$\begin{aligned} \triangle_\Sigma r^2 &= \triangle_\Sigma \langle x, x\rangle \\ &= 2\langle \triangle_\Sigma x, x\rangle + 2|\nabla x|^2, \end{aligned}$$

where ∇ and $\triangle_\Sigma$ are the Riemannian connection and the Laplacian of Σ, respectively. Since $|\nabla x|^2 = 2$ and $\triangle_\Sigma x = 2H\nu$, the following holds.

$$\triangle_\Sigma r^2 = 4\left(1 + H\langle x, \nu\rangle\right). \tag{2.1}$$

On the other hand, integrating the left hand side of (2.1) on Σ, we have

$$\begin{aligned}\int_\Sigma \triangle_\Sigma r^2 \, dA &= \int_{\partial\Sigma} -2r\frac{\partial r}{\partial \eta} \, ds \\ &= \int_{\partial\Sigma} 2r\,\langle \nabla r, x\rangle \, ds \\ &= \int_{\partial\Sigma} 2 \, ds \end{aligned} \tag{2.2}$$

by using the divergence theorem. In the last equality, the facts that $r = 1$ and ∇r is parallel to x on $\partial\Sigma$ are used. From (2.1) and (2.2),

$$\begin{aligned} L(\partial\Sigma) &= \frac{1}{2}\int_\Sigma \triangle_\Sigma r^2 \, dA \\ &= 2\int_\Sigma (1 + H\langle x, \nu\rangle) \, dA. \end{aligned}$$

We get the conclusion. □

3. A gap theorem of Simons' type

Let Σ be a cmc-H surface in $\mathbb{R}^3$. The traceless second fundamental form $\mathring{A}$ is defined to be

$$\mathring{A} = A - H \circ g_\Sigma,$$

where g_Σ is the induced metric on Σ. The squared norm of the traceless second fundamental form is easily computed in terms of that of the second fundamental form such that

$$|\mathring{A}|^2 = (\lambda_1 - H)^2 + (\lambda_2 - H)^2 = |A|^2 - 2H^2.$$

If Σ is minimal, then $\mathring{A} = A$. We say that a point $p \in \Sigma$ is a *umbilical point* if $\mathring{A}(p) = 0$, which is equivalent to $\lambda_1 = \lambda_2$ at p. It is well-known that either the set of umbilical points of a cmc-H surface Σ is isolated, or Σ is totally umbilical, i.e., every point of Σ is a umbilical point. Moreover, if Σ is a totally umbilical cmc-H surface in $\mathbb{R}^3$, then Σ is part of a totally geodesic plane when $H = 0$, or is part of a round sphere when $H \neq 0$, respectively. For any free boundary cmc-H surface in the unit ball, the following integral equality holds.

PROPOSITION 3.1. *Let Σ be a free boundary cmc-H surface in the unit ball in $\mathbb{R}^3$. Then*

$$\int_\Sigma |\mathring{A}|^2 \, dA = \int_\Sigma \left(2H^2 + 4\left(1 + H\langle x, \nu\rangle\right)\right) \, dA - 4\pi\chi(\Sigma), \tag{3.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Proof. The squared norm of the second fundamental form is given by

$$|A|^2 = \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = 4H^2 - 2K. \tag{3.2}$$

From Lemma 2.2 and Lemma 2.3,

$$\int_{\partial\Sigma} \kappa_g \, ds = L(\partial\Sigma) = 2\int_\Sigma (1 + H\langle x, \nu\rangle) \, dA.$$

By using the equation (3.2), we obtain

$$\int_\Sigma K \, dA = \frac{1}{2}\int_\Sigma \left(4H^2 - |A|^2\right) \, dA = \frac{1}{2}\int_\Sigma \left(2H^2 - |\mathring{A}|^2\right) \, dA.$$

Therefore with the aid of the Gauss-Bonnet formula, the following equality holds:

$$\int_\Sigma \left(2H^2 - |\mathring{A}|^2\right) \, dA + 4\int_\Sigma (1 + H\langle x, \nu\rangle) \, dA = 4\pi\chi(\Sigma).$$

Arranging the expression, we get the conclusion. □

Proposition 3.2. *Let Σ be a free boundary cmc-H surface in the unit ball in $\mathbb{R}^3$. Assume $|\mathring{A}|^2$ is constant on Σ. Then Σ is totally umbilical.*

Proof. The famous Simons' identity for a cmc-H surface in $\mathbb{R}^3$ (see [22]) is as follows.

$$\triangle_\Sigma |\mathring{A}| - \frac{|\nabla|\mathring{A}||^2}{|\mathring{A}|} + \left(|\mathring{A}|^2 - 2H^2\right)|\mathring{A}| = 0.$$

If $|\mathring{A}|^2$ is constant, then $\left(|\mathring{A}|^2 - 2H^2\right)|\mathring{A}| = 0$. It follows that either Σ is a totally umbilical, or $K \equiv 0$ and therefore each principal curvature is constant. As consequences, Σ is a right angular cylinder unless it is totally umbilical. But a right angular cylinder does not satisfy the free boundary condition. This proves the Proposition 3.2. □

The main theorem is a gap result for free boundary cmc-H surfaces in the unit ball. It can be thought as that for free boundary cmc-H surfaces analogous to a gap theorem for closed minimal surfaces in $\mathbb{S}^3$ by Simons. But except a totally umbilical surface, $|\mathring{A}|^2$ cannot be constant on free boundary cmc-H surfaces in $\mathbb{R}^3$ unlike as in $\mathbb{S}^3$.

Theorem 3.3. *Let Σ be a free boundary cmc-H surface in the unit ball in $\mathbb{R}^3$. Then*

- *either* $|\mathring{A}|^2 \equiv 0$ *and* Σ *is totally umbilical, i.e.,* Σ *is an equatorial disk if* $H = 0$ *and a spherical cap if* $H \neq 0$*, respectively;*
- *or* $|\mathring{A}|^2(x) \geq 2H^2 + 4 + 4H\langle x, \nu(x)\rangle$ *at some point* $x \in \Sigma$*, furthermore,* $|A|^2(x) > 4$ *at some point* $x \in \Sigma$ *if* $H = 0$*.*

Proof. Note that Σ is a surface with boundary Therefore $\chi(\Sigma) \leq 1$. Suppose that $\chi(\Sigma) = 1$. Then Σ is a topological disk. Remind that the only free boundary cmc-H surface in the unit ball is part of a equatorial disk or part of a spherical cap by Nitsche [21], and hence $\mathring{A} \equiv 0$. Now we may assume that $\chi(\Sigma) \leq 0$. From the equation (3.1), an integral inequality holds as follows.

$$\int_\Sigma |\mathring{A}|^2 \, dA \geq \int_\Sigma \left(2H^2 + 4 + 4H\langle x, \nu\rangle\right) \, dA. \tag{3.3}$$

Suppose that $|\mathring{A}|^2(x) < 2H^2 + 4 + 4H\langle x, \nu(x)\rangle$ for any point $x \in \Sigma$. Integrating both sides, we have

$$\int_\Sigma |\mathring{A}|^2 \, dA < \int_\Sigma \left(2H^2 + 4 + 4H\langle x, \nu\rangle\right) \, dA.$$

Comparing with the inequality (3.3), we get a contradiction. To prove the remaining part, let Σ be a minimal surface. Suppose that $|A|^2(x) \leq 4$ for all $x \in \Sigma$. Then $|A|^2 \equiv 4$ on Σ. Since $|A|^2 = 2\lambda_1^2 = 2\lambda_2^2$, the curvature κ of $\partial\Sigma$ is given by

$$\kappa^2 = \kappa_n^2 + \kappa_g^2 = \lambda_1^2 + 1 = 3$$

because $\partial\Sigma$ is a line of curvature of Σ. And therefore, each component of $\partial\Sigma$ is a circle in a plane which meets Σ at a constant contact angle along $\partial\Sigma$. Any immersed minimal surface meets a plane at a constant contact angle along a circle is part of a catenoid (see [23]). But $|A|^2$ is not constant on a catenoid. It is a contradiction. $\square$

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